

# NOVIKOV OPERAD IS NOT KOSZUL

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ABSTRACT. An algebra with identities  $a \circ (b \circ c - c \circ b) = (a \circ b) \circ c - (a \circ c) \circ b$  and  $a \circ (b \circ c) = b \circ (a \circ c)$  is called Novikov. We show that Novikov operad is not Koszul.

## 1. INTRODUCTION

After putting my paper [3] on ArXive I receive several questions concerning non-Koszulity of Novikov operad. In [3] I refer to [4] to non-Koszulity of Novikov operad. It is mistake. I am gratefull to P. Zusmanovich and V.Dotsenko who have noticed that the paper [4] does not contain non-Koszulity of Novikov operad. By these reasons I am putting on ArXive proof of this statement. We will save notations of [3] .

Recall that an algebra  $(A, \circ)$  is called *right-symmetric* if it satisfies the identity

$$(a, b, c) = (a, c, b), \quad \forall a, b, c \in A,$$

where

$$(a, b, c) = a \circ (b \circ c) - (a \circ b) \circ c$$

is associator. Algebra is *left-commutative*, if

$$a \circ (b \circ c) = b \circ (a \circ c),$$

for any  $a, b, c \in A$ . Similarly, an algebra with identity

$$(a, b, c) = (b, a, c)$$

is *left-symmetric*. The identity

$$(a \circ b) \circ c = (a \circ c) \circ b$$

is called it right-commutative identity.

A right-commutative left-symmetric algebra  $(A, \circ)$  is called *left-Novikov*. A left-commutative right-symmetric algebra is called *right-Novikov*. About Novikov algebras see [1], [5], [7].

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**Proposition 1.1.** *For any left-(right-)Novikov algebra  $(A, \circ)$  a new algebra  $(A, \star)$ , where  $a \star b = b \circ a$  is an opposite multiplication, became right-(left-)Novikov.*

**Corollary 1.2.** *Let  $F_n^r$  be a multilinear part of free right-Novikov algebra generated by  $n$  elements and similarly let  $F_n^l$  be a multilinear part of free left-Novikov algebra generated by  $n$  elements. Then  $\dim F_n^r = \dim F_n^l$ .*

The aim of our paper is to prove the following result.

**Theorem 1.3.** *Dual operad to left-(right-)Novikov operad is right-(left-)Novikov. Novikov operad is not Koszul.*

**Proof.** Free magma has 12 multilinear elements of degree 3. For a free Novikov algebras only 6 of them form base. Let us construct multilinear base elements of degree 3 for free Novikov algebras. Bases of free Novikov algebras in terms of rooted trees and so-called  $r$ -elements was constructed in [2]. We use free Novikov base construction by Young diagrams given in [3].

Young diagrams of degree 2:



Novikov diagrams of degree 3:



Novikov tableaux of degree 3 generated by elements  $a, b, c$ :

$$\begin{array}{cc} b & c \\ a & \end{array} \quad \begin{array}{cc} c & b \\ a & \end{array} \quad \begin{array}{cc} c & a \\ b & \end{array} \quad \begin{array}{ccc} a & b & c \end{array} \quad \begin{array}{ccc} b & a & c \end{array} \quad \begin{array}{ccc} c & a & b \end{array}$$

Multilinear base elements of degree 3:

$$a \circ (b \circ c), \quad a \circ (c \circ b), \quad c \circ (a \circ b), \quad (a \circ b) \circ c, \quad (b \circ a) \circ c, \quad (c \circ a) \circ b.$$

Below we give presentation of 6 non-base elements of degree 3 as a linear combination of above constructed 6 base elements of degree 3:

$$\begin{aligned} b \circ (a \circ c) &= a \circ (b \circ c), \quad c \circ (a \circ b) = a \circ (c \circ b), \quad c \circ (b \circ a) = b \circ (c \circ a), \\ (a \circ c) \circ b &= (a \circ b) \circ c + a \circ (c \circ b) - a \circ (b \circ c), \\ (b \circ c) \circ a &= -a \circ (b \circ c) + b \circ (c \circ a) + (b \circ a) \circ c, \\ (c \circ b) \circ a &= (c \circ a) \circ b - a \circ (c \circ b) + b \circ (c \circ a). \end{aligned}$$

Then

$$\begin{aligned}
& [[a \otimes u, b \otimes v], c \otimes w] \\
&= ((a \circ b) \circ c) \otimes ((u \cdot v) \cdot w) - ((b \circ a) \circ c) \otimes ((v \cdot u) \cdot w) - (c \circ (a \circ b)) \otimes (w \cdot (u \cdot v)) \\
&\quad + (c \circ (b \circ a)) \otimes (w \cdot (v \cdot u)) \\
&= ((a \circ b) \circ c) \otimes ((u \cdot v) \cdot w) - ((b \circ a) \circ c) \otimes ((v \cdot u) \cdot w) \\
&\quad - (a \circ (c \circ b)) \otimes (w \cdot (u \cdot v)) + (b \circ (c \circ a)) \otimes (w \cdot (v \cdot u)) \\
&= ((a \circ b) \circ c) \otimes ((u \cdot v) \cdot w) + ((b \circ a) \circ c) \otimes (-(v \cdot u) \cdot w) \\
&\quad + (a \circ (c \circ b)) \otimes (-w \cdot (u \cdot v)) + (b \circ (c \circ a)) \otimes (w \cdot (v \cdot u)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
& [[b \otimes v, c \otimes w], a \otimes u] \\
&= ((b \circ c) \circ a) \otimes ((v \cdot w) \cdot u) - ((c \circ b) \circ a) \otimes ((w \cdot v) \cdot u) \\
&\quad - (a \circ (b \circ c)) \otimes (u \cdot (v \cdot w)) + a \circ (c \circ b) \otimes (u \cdot (w \cdot v)) \\
&= (-a \circ (b \circ c) + b \circ (c \circ a) + (b \circ a) \circ c) \otimes ((v \cdot w) \cdot u) \\
&\quad + (-(c \circ a) \circ b + a \circ (c \circ b) - b \circ (c \circ a)) \otimes ((w \cdot v) \cdot u) \\
&\quad - (a \circ (b \circ c)) \otimes (u \cdot (v \cdot w)) + (a \circ (c \circ b)) \otimes (u \cdot (w \cdot v)) \\
&= (a \circ (b \circ c)) \otimes (-(v \cdot w) \cdot u) + (b \circ (c \circ a)) \otimes ((v \cdot w) \cdot u) + ((b \circ a) \circ c) \otimes ((v \cdot w) \cdot u) \\
&\quad + (-(c \circ a) \circ b + a \circ (c \circ b) - b \circ (c \circ a)) \otimes ((w \cdot v) \cdot u) \\
&\quad - (a \circ (b \circ c)) \otimes (u \cdot (v \cdot w)) + (a \circ (c \circ b)) \otimes (u \cdot (w \cdot v)).
\end{aligned}$$

Further,

$$\begin{aligned}
& [[c \otimes w, a \otimes u], b \otimes v] \\
&= ((c \circ a) \circ b) \otimes ((w \cdot u) \cdot v) - ((a \circ c) \circ b) \otimes ((u \cdot w) \cdot v) - (b \circ (c \circ a)) \otimes (v \cdot (w \cdot u)) \\
&\quad + (b \circ (a \circ c)) \otimes (v \cdot (u \cdot w)) \\
&= ((c \circ a) \circ b) \otimes ((w \cdot u) \cdot v) \\
&\quad + (-(a \circ b) \circ c - a \circ (c \circ b) + a \circ (b \circ c)) \otimes ((u \cdot w) \cdot v) \\
&\quad + (b \circ (c \circ a)) \otimes (-v \cdot (w \cdot u)) \\
&\quad + (a \circ (b \circ c)) \otimes (v \cdot (u \cdot w))
\end{aligned}$$

$$= ((c \circ a) \circ b) \otimes ((w \cdot u) \cdot v) + ((a \circ b) \circ c) \otimes (-(u \cdot w) \cdot v) + (a \circ (c \circ b)) \otimes (-(u \cdot w) \cdot v) + a \circ (b \circ c) \otimes ((u \cdot w) \cdot v) + (b \circ (c \circ a)) \otimes (-v \cdot (w \cdot u)) + (a \circ (b \circ c)) \otimes (v \cdot (u \cdot w)).$$

Therefore,

$$\begin{aligned} & [[a \otimes u, b \otimes v], c \otimes w] + [[b \otimes v, c \otimes w], a \otimes u] + [[c \otimes w, a \otimes u], b \otimes v] \\ &= ((a \circ b) \circ c) \otimes \{(u \cdot v) \cdot w - (u \cdot w) \cdot v\} \\ &+ (a \circ (b \circ c)) \otimes \{-(v \cdot w) \cdot u - u \cdot (v \cdot w) + v \cdot (u \cdot w) + (u \cdot w) \cdot v\} \\ &+ (a \circ (c \circ b)) \otimes \{-w \cdot (u \cdot v) + (w \cdot v) \cdot u + u \cdot (w \cdot v) - (u \cdot w) \cdot v\} \\ &+ ((b \circ a) \circ c) \otimes \{-(v \cdot u) \cdot w + (v \cdot w) \cdot u\} \\ &+ (b \circ (c \circ a)) \otimes \\ &\{+w \cdot (v \cdot u) + (v \cdot w) \cdot u - (w \cdot v) \cdot u - v \cdot (w \cdot u)\} \\ &+ ((c \circ a) \circ b) \otimes \{-(w \cdot v) \cdot u + (w \cdot u) \cdot v\}. \end{aligned}$$

Thus the Lie-admissibility condition for  $A \otimes U$ , where  $A$  is free right-Novikov algebra, is equivalent to the following conditions

$$\begin{aligned} & (u \cdot v) \cdot w - (u \cdot w) \cdot v = 0, \\ & -(v \cdot w) \cdot u - u \cdot (v \cdot w) + v \cdot (u \cdot w) + (u \cdot w) \cdot v = 0, \\ & -w \cdot (u \cdot v) + (w \cdot v) \cdot u + u \cdot (w \cdot v) - (u \cdot w) \cdot v = 0, \\ & -(v \cdot u) \cdot w + (v \cdot w) \cdot u = 0, \\ & w \cdot (v \cdot u) + (v \cdot w) \cdot u - (w \cdot v) \cdot u - v \cdot (w \cdot u) = 0, \\ & -(w \cdot v) \cdot u + (w \cdot u) \cdot v = 0. \end{aligned}$$

Note that all of these identities are consequences of left-symmetric and right-commutative identities. So,  $(U, \cdot)$  is left-Novikov, if  $(A, \circ)$  is right-Novikov. Similarly,  $(U, \cdot)$  is right-Novikov, if  $(A, \circ)$  is left-Novikov. These mean that dual operad to right-(left-)Novikov operad is left-(right-)Novikov operad.

We have noted that categories of left-Novikov and right-Novikov algebras are equivalent. If we change left-Novikov multiplication to opposite multiplication we obtain right-Novikov multiplication. By Theorem 1.1 [3] and by corollary 1.2 of proposition 1.1 dimensions of multilinear parts of free left-(right-)Novikov algebras have dimensions 1, 2, 6, 20, 70 for degrees 1, 2, 3, 4, 5. Therefore, beginning parts of Hilbert series of Novikov and dual Novikov operads looks like

$$H(t) = H^1(t) = -t + t^2 - t^3 + 20t^4/24 - 70t^5/120 + O(t^6).$$

Thus,

$$H(H^1(t)) = t + t^5/6 + O(t^6) \neq t.$$

So, by results of [6] left-(right-)Novikov operad is not Koszul. Theorem 1.3 is proved.

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